Evolution of the trans-Alfvénic normal shock in a gas of finite electrical conductivity

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(Received 26 June 1963)

The stability of a steady, plane, one-dimensional trans-Alfvénic shock to small transverse disturbances in the velocity and magnetic field is discussed. In the undisturbed flow the magnetic field and velocity are normal to the plane of the shock which is treated as a discontinuity in an inviscid gas of zero thermal conductivity. However, the electrical conductivity of the ambient gas is taken to be finite, i.e. the magnetic diffusivity is very much larger than both the viscous diffusivity and the thermal diffusivity.

A small uniform transverse perturbation of the magnetic field is imposed at zero time. A specific super-Alfvénic normal shock is chosen. Computed values of the magnetic field, in and near the shock, are given for various later times. Two diffusing Alfvén waves of the amplitudes predicted by infinite conductivity theory are shown to propagate away from the shock region, leaving behind the expected steady-state shock profile.

Similar computations are carried out for a specific trans-Alfvénic normal shock. An analytic asymptotic solution, valid for large times, is also obtained. This result agrees with the computations carried out. The shock profile of the transverse quantities is one which grows linearly with time. Outside this magnetohydrodynamic shock region which surrounds the trans-Alfvénic hydrodynamic shock (discontinuity), steady states are reached in each of which the transverse velocity and transverse magnetic field are uniform. An incident Alfvén wave, consisting of a weak diffusing current sheet, produces these same effects. The resolution of an arbitrary transverse fluctuation, in which both the magnetic field and the velocity have limiting values at large distances from the shock, is discussed. The solution for large times is found.

It is shown that the trans-Alfvénic normal shock and its two limiting cases, the null switch-on and null switch-off shocks, are unstable to general transverse disturbances although there exist particular disturbances of this kind which will not destroy them. An integral condition is obtained which, together with the relevant boundary conditions, determines the profiles of the transverse quantities in the trans-Alfvénic normal shock whenever a steady state is reached. This removes the puzzling arbitrariness of these profiles.

1. Introduction

A great deal of work has been done in recent years on the stability of steady, plane, one-dimensional magnetohydrodynamic shocks. These shocks were treated as discontinuities in a perfect fluid. Before we proceed with a brief discussion of this work, it is expedient to introduce some of the terminology which will be used.

A Cartesian set of axes, $Oxyz$, is employed. The x-axis points downstream, in the direction of variation. This is normal to the plane of the shock and is also referred to as the longitudinal direction. Contact surfaces and tangential discontinuities (Landau & Lifshitz 1960) are not considered as shocks. Except in the case of the Alfvén† shock, Oyz can be chosen (e.g. Shercliff 1960*b*) moving in the shock, such that the z-components of the magnetic field and velocity are zero. (A more precise statement is that the z-component of the magnetic field, B_z, is zero *outside the shock region*. In spite of this restriction, there are certain magnetohydrodynamic shocks within which B_z can take small, but non-zero, values, i.e. have a shock profile. This statement will be amplified later.) A magnetohydrodynamic shock is termed transverse, oblique or normal, according as the undisturbed magnetic field $(B_x, B_y, 0)$ relative to this system of axes, is parallel to, inclined to, or normal to the plane of the shock. The Alfvén shock, to which the last statement does not apply, involves only an arbitrary rotation of the transverse magnetic field and velocity. It is stable (Syrovatskii 1953) and is omitted except in so far as it is a limiting case of other types of shock.

The longitudinal Alfvén velocity is the velocity with which small transverse disturbances in the z-direction are propagated, relative to the fluid. The Alfvén number m is the ratio of the normal velocity u to the longitudinal Alfvén velocity. The upstream region is referred to as region 1 and all quantities evaluated there are given the suffix 1, and similarly for the downstream region, region 2. Since magnetohydrodynamic shocks are never expansive (Iordanskii 1958; Shercliff 1960*a*) m_1 is never less than m_2. In this paper an oblique or normal shock is termed super-Alfvénic, trans-Alfvénic or sub-Alfvénic accordingly as $m_2 > 1$, $m_1 > 1 > m_2$ or $1 > m_1$. Under this classification the so-called fast oblique shocks (see, for example, Shercliff 1960*b*) are super-Alfvénic and the so-called slow oblique shocks are sub-Alfvénic. The intermediate oblique shocks are trans-Alfvénic. A shock is called a switch-on shock if $m_1 > m_2 = 1$ and $(B_y)_1 = 0 \neq (B_y)_2$. If $(B_y)_2 = 0$, we shall call it a null switch-on shock. A switch-off shock has $m_1 = 1 > m_2$ and $(B_y)_1 \neq 0 = (B_y)_2$. If $(B_y)_1 = 0$, we shall label it a null switch-off shock.

In all the papers mentioned in the following paragraph, shocks are treated as discontinuities and the direction of variation of the one-dimensional disturbances is normal to the plane of the shock. Also, since the gas is treated as having negligible electrical resistance the general perturbation field is composed of the 14 possible plane waves. The authors declare a shock to be stable if, and only if, satisfaction of the boundary conditions at the shock uniquely determines the amplitudes of the outgoing waves in terms of the amplitudes of the incoming waves. If these amplitudes are not uniquely determined, then the shock can emit waves spontaneously, implying instability. For some shocks, the boundary conditions overdetermine the amplitudes and there is no solution at all. The trans-Alfvénic shock is in this category. The effects of electrical resistance must be considered, and this is one of the objects of the present paper.

† This 'shock' is in fact a simple wave.

Akhiezer, Liubarskii & Polovin (1958) showed that transverse shocks are stable. Syrovatskii (1959) proved that all trans-Alfvénic shocks are unstable to small perturbations in the z-direction and that super-Alfvénic and sub-Alfvénic shocks are stable to such disturbances. Akhiezer *et al.* also found that fast oblique and slow oblique shocks are stable to a general small disturbance but that intermediate oblique shocks are not. They showed that some species of these intermediate oblique shocks have instabilities in addition to the one found by Syrovatskii. Lastly, they proved that a normal shock is unstable only if it is trans-Alfvénic, null switch-on or null switch-off. Such shocks were found to be unstable to transverse disturbances. Syrovatskii (1959) and Polovin (1961) pointed out that the trans-Alfvénic normal shock can be replaced by a switch-on shock followed, at an indeterminate distance downstream, by a switch-off shock. Indeed a rotational Alfvén wave could be superimposed in between these two shocks. However, diffusion prevents this latter wave existing in a steady state and, as Syrovatskii pointed out, both the switch-on and switch-off shocks are unstable. Thus no headway is made by proposing these models.

Anderson (1963) has summarized the work of Kontorovich (1959) who showed that the conclusions obtained on stability with respect to normal small disturbances hold for arbitrary small disturbances. Indeed Kontorovich found that one of the species of intermediate shocks is unstable to normal disturbances only. Hence normal disturbances are the most severe ones.

This discussion of previous work indicates the desirability of classifying shocks as super-, trans- or sub-Alfvénic, as has been done. It is worth mentioning at this point that v_z and B_z, subject to their remaining small, can have an arbitrary shock profile in all steady, trans-Alfvénic shocks.

This paper discusses the stability of the trans-Alfvénic normal shock to small, normal disturbances. The initial value problem is discussed when the electrical conductivity of the ambient gas is finite and is a scalar quantity in each of the flow regions, i.e. the flow is collision dominated and the electron inertia term, as well as the electron pressure gradient term, is neglected compared to the retained terms in the generalized Ohm's Law. Thus in contrast to previous works there is now a natural length scale available, namely the magnetohydrodynamic shock thickness. This is an entirely realistic situation. Moreover, for this problem continuum theory is valid.

2. Equations and boundary conditions

2.1. *Full equations and boundary conditions*

The magnetohydrodynamic equations governing the one-dimensional, unsteady motion of an inviscid, electrically conducting, compressible fluid of zero thermal conductivity are (in M.K.S. units)

$$\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{2\mu\rho}\frac{\partial}{\partial x}(B_y^2+B_z^2) = 0, \tag{1}$$

$$\frac{\partial \rho}{\partial t}+V_x\frac{\partial \rho}{\partial x}+\rho\frac{\partial V_x}{\partial x} = 0, \tag{2}$$

$$\rho\theta\frac{\partial s}{\partial t}+\rho\theta V_x\frac{\partial s}{\partial x} = \rho\frac{\partial e}{\partial t}+\rho V_x\frac{\partial e}{\partial x}-\frac{p}{\rho}\frac{\partial \rho}{\partial t}-\frac{p}{\rho}V_x\frac{\partial \rho}{\partial x} = \mu^{-1}\lambda\left\{\left(\frac{\partial B_y}{\partial x}\right)^2+\left(\frac{\partial B_z}{\partial x}\right)^2\right\}, \quad (3)$$

$$B_x = \text{const.}, \quad (4)$$

$$\frac{\partial B_q}{\partial t}+V_x\frac{\partial B_q}{\partial x}-\frac{\partial}{\partial x}\left(\lambda\frac{\partial B_q}{\partial x}\right)-B_x\frac{\partial V_q}{\partial x}+B_q\frac{\partial V_x}{\partial x} = 0 \quad (q = y,z), \quad (5)$$

$$\rho\frac{\partial V_q}{\partial t}+\rho V_x\frac{\partial V_q}{\partial x}-\frac{B_x}{\mu}\frac{\partial B_q}{\partial x} = 0 \quad (q = y,z). \quad (6)$$

The notation is standard, λ being the magnetic diffusivity and μ the permeability of the fluid. The entropy per unit mass s, internal energy e and absolute temperature θ are known functions of p and ρ. Hence only two of θ, e, p, ρ and s are independent. A consequence of equation (4) is that $m_1^2 = Zm_2^2$ where $Z = \rho_2/\rho_1$ is the shock density ratio. The current $\mathbf{j}$ and electric field $\mathbf{E}$ are given in terms of $\mathbf{B}$ and $\mathbf{v}$ by the equations

$$\mathbf{j} = \frac{1}{\mu}\mathbf{i}\wedge\frac{\partial \mathbf{B}}{\partial x}, \quad (7)$$

$$\mathbf{E} = \lambda\mu\mathbf{j}-\mathbf{v}\wedge\mathbf{B} \quad (8)$$

where $\mathbf{i}$ is a unit vector parallel to Ox.

The full, non-linearized, boundary conditions satisfied at a steady, plane, one-dimensional, shock propagating through the fluid are given below. $[A]$ denotes the change in A across the shock.

$$[p+\rho V_x^2+(B_y^2+B_z^2)/2\mu] = 0, \quad (9)$$

$$[\rho V_x] = 0, \quad (10)$$

$$[e+p/\rho+(\mathbf{E}\wedge\mathbf{B})_x/\mu\rho V_x+\tfrac{1}{2}(V_x^2+V_y^2+V_z^2)] = 0, \quad (11)$$

$$\left[V_x B_q-\lambda\frac{\partial B_q}{\partial x}-B_x V_q\right] = 0, \quad (12)$$

$$\left[\rho V_x V_q-\frac{B_x}{\mu}B_q\right] = 0, \quad (13)$$

$$[B_q] = 0. \quad (14)$$

Equations (1) to (6). with the appropriate suffix, govern the flow in each of the regions separated by the shock. The boundary conditions (9) to (13) have not been simplified in the obvious manner because (14) does not necessarily hold when $\lambda = 0$.

2.2. *Linearized equations*

Let us now use these equations to discuss the stability of such a shock to small perturbations in the quantities concerned. It is supposed that in the unperturbed system the density, pressure and longitudinal velocity are constant in each region, $B_{q_1} = B_{q_2} = 0$, $\mathbf{j}_1 = \mathbf{j}_2 = 0$ and $V_{q_1} = V_{q_2} = 0$ $(q = y,z)$.

As was pointed out by Akhiezer *et al.*, albeit for the case $\lambda = 0$, the relevant linearized equations and boundary conditions split up into three separate sets. The first set involves the changes in pressure, density, longitudinal velocity relative to the stationary axes, and the velocity of the shock front. This is the well known stability problem relevant to the case of the shock propagating through a non-conducting gas. Hence we need concern ourselves no further with it. The shock is stable to such perturbations if $M_1 > 1$ and $M_2 < 1$, inequalities that are always satisfied.

The other two sets are of identical form for $q = y, z$, and need no separate discussion. They are

$$\frac{\partial B_{qi}}{\partial t} + u_i \frac{\partial B_{qi}}{\partial x} - \lambda_i \frac{\partial^2 B_i}{\partial x^2} - B_x \frac{\partial V_{q1}}{\partial x} = 0, \tag{15}$$

$$\rho_i \frac{\partial V_{qi}}{\partial t} + \rho_i u_i \frac{\partial V_{qi}}{\partial x} - \frac{B_x}{\mu} \frac{\partial B_{qi}}{\partial x} = 0, \tag{16}$$

where the suffix $i = 1, 2$ is used to label conditions upstream and downstream of the shock, and the boundary conditions (12), (13) and (14). In (15) and (16), u is the velocity perpendicular to the shock in the undisturbed flow, and ρ and λ have their undisturbed values. The suffix q will be retained and can be read as representing either y or z. Terms of second order in the perturbed quantities have been everywhere neglected.

Akhiezer *et al.* discussed the resolution of small disturbances in B_q and v_q but set $\lambda = 0$. In this case, equations (15) and (16) yield the well-known Alfvén wave equation. The boundary conditions obeyed were taken as (12) (with $\lambda = 0$) and (13). It is important for the reader to realize that this last statement requires that the integral of $\rho(\partial V_q/\partial t)$ and of $\partial B_q/\partial t$, over a box-type control volume of dimensions $(\delta, 1, 1)$, surrounding unit area of the shock, should become negligible compared to the retained terms, as δ tends to the width of the magnetohydrodynamic shock. (It is shown later that this is not always true.)

If $m_1^2 > Z$ or $m_1^2 < 1$, there are two Alfvén waves travelling outward from the shock and the two boundary conditions determine these uniquely in terms of the two incoming Alfvén waves. If, however, $Z > m_1^2 > 1$, only one of the four waves is outward travelling and it is impossible in general to satisfy the boundary conditions. In this paper, these problems will be tackled with λ not set equal to zero.

2.3. *Linearized equations and boundary conditions in non-dimensional form*

Let us define the following non-dimensional quantities:

$$b_i = B_{qi}/B_x, \quad v_i = V_{qi}/u_i, \quad X_i = u_i x/\lambda_i, \quad T_i = u_i^2 t/\lambda_i, \quad Q_1 = T_2/T_1 = Q, \quad Q_2 = 1.$$

The suffices will be omitted from the X's as it is always obvious which is appropriate. T_2 will simply be written as T. Equations (12) to (16) may now be rewritten as follows:

$$\left(Q_i \frac{\partial}{\partial T} + \frac{\partial}{\partial X} - \frac{\partial^2}{\partial X^2}\right) b_i - \frac{\partial v_i}{\partial X} = 0, \tag{17}$$

$$\frac{\partial b_i}{\partial X} - m_i^2 \left(Q_i \frac{\partial}{\partial T} + \frac{\partial}{\partial X}\right) v_i = 0. \tag{18}$$

The boundary conditions require that, at $X = 0$,

$$b_1 = b_2, \tag{19}$$

$$Z v_1 = v_2, \tag{20}$$

and

$$(Z-1)\, b = Z \frac{\partial b_1}{\partial X} - \frac{\partial b_2}{\partial X}. \tag{21}$$

2.4. *Steady state*

It is desirable at this stage to use the steady-state equations to discuss the variation of b, v through the magnetohydrodynamic shock when it has end states connected by the boundary conditions (12) and (13) with $\lambda = 0$.

We obtain
$$(b_2)_{+\infty}/(b_1)_{-\infty} = Z(m_1^2-1)/(m_1^2-Z), \tag{22}$$
$$(v_2)_{+\infty} = Z(v_1)_{-\infty}+Z(Z-1)(b_1)_{-\infty}/(m_1^2-Z). \tag{23}$$

(*a*) $Z < m_1^2$, *i.e. super-Alfvénic normal shock*

In this case the solutions are
$$b_1/(b_1)_{-\infty} = 1+\frac{m_1^2(Z-1)}{m_1^2-Z}\exp(1-m_1^{-2})X_1, \tag{24}$$
$$v_1 = (v_1)_{-\infty}+\frac{(Z-1)(b_1)_{-\infty}}{m_1^2-Z}\exp(1-m_1^{-2})X_1, \tag{25}$$
and v_2 and b_2 are constant, i.e. $v_2 = (v_2)_{+\infty}$ and $b_2 = (b_2)_{+\infty}$.

(*b*) $m_1^2 < 1$.

In this case, b_1 and v_1 are constant, and
$$b_2/(b_2)_{+\infty} = 1+\frac{m_1^2(Z-1)}{Z(1-m_1^2)}\exp\left\{\frac{-(Z-m_1^2)X}{m_1^2}\right\}, \tag{26}$$
$$v_2 = (v_2)_{+\infty}+(b_2)_{+\infty}\frac{(Z-1)}{(1-m_1^2)}\exp\left\{\frac{-(Z-m_1^2)X}{m_1^2}\right\}. \tag{27}$$

(*c*) $Z > m_1^2 > 1$

This is by far the most interesting case. The solutions are of the form
$$b_1 = (b_1)_{-\infty}+A\exp(1-m_1^{-2})X, \tag{28}$$
$$v_1 = (v_1)_{-\infty}+Am_1^{-2}\exp(1-m_1^{-2})X, \tag{29}$$
$$b_2 = (b_2)_{+\infty}-\frac{[m_1^2(Z-1)(b_1)_{-\infty}+(Z-m_1^2)A]}{(m_1^2-Z)}\exp\left\{\frac{-(Z-m_1^2)X}{m_1^2}\right\}, \tag{30}$$
and
$$v_2 = (v_2)_{+\infty}-\frac{[m_1^2(Z-1)(b_2)_{+\infty}+Z(m_1^2-1)A]}{m_1^2(m_1^2-1)}\exp\left\{\frac{-(Z-m_1^2)X}{m_1^2}\right\}. \tag{31}$$

These results satisfy the differential equations and the boundary conditions for *any* numerical constant A. Hence apparently all trans-Alfvénic normal shocks have indeterminate structure. We shall see later that, when in fact a steady state exists, A is determined from previous events by an integral expression.

3. The initial value problem

Let us now study the following Cauchy-type problem. Given that, at $T = 0$, $b = b(x,0)$, $v = v(x,0)$, what is the history of the subsequent changes in b and v? In terms of the Laplace transforms
$$\xi_i = \int_0^\infty b_i e^{-sT}dT \quad \text{and} \quad \eta_i = \int_0^\infty v_i e^{-sT}dT,$$

where $s = \phi + i\psi$ and $\phi > 0$, equations (17) and (18) become

$$\left(\frac{d^2}{dX^2} - \frac{d}{dX} - Q_i s\right)\xi_i + \frac{d\eta_i}{dX} = -Q_i b_i(X, 0), \tag{32}$$

$$\frac{d\xi_i}{dX} - m_i^2\left(\frac{d}{dX} + Q_i s\right)\eta_i = -m_i^2 Q_i v_i(X, 0). \tag{33}$$

Hence
$$\xi_i = \sum_{p=1}^{3} A_{ip} \exp\{\beta_{ip} X\} + (\text{particular integral}),$$

where the β_{ip} are the roots of

$$\beta^3 + (m_i^{-2} - 1 + Q_i s)\beta^2 - 2Q_i s\beta - Q_i^2 s^2 = 0. \tag{34}$$

The A_{ip} are constants depending on the β's, Q, m_1, Z and s. It can be shown that one root of this cubic has a positive real part and that two roots have a negative real part, for all values of m_i^2, Q_i and s, provided that $\mathrm{Re}(s) > 0$. Thus the condition that ξ_i remains finite as $|X| \to \infty$ rules out three of the arbitrary constants. As we have three boundary conditions to satisfy, the problem is solvable for all m_1^2, Z and Q. These boundary conditions are obtained from equations (19) to (21). They are that, at

$$X = 0, \quad \xi_1 = \xi_2, \quad Z\eta_1 = \eta_2 \quad \text{and} \quad (Z-1)\xi = \left(Z\frac{\partial \xi_1}{\partial X} - \frac{\partial \xi_2}{\partial X}\right).$$

3.1. *The super-Alfvénic case, i.e. $m_1^2 > Z$*

The initial disturbance is $b(x, 0) = \Delta$ (const.) and $v(x, 0) = 0$, all X. The asymptotic forms of the solutions, as $T \to \infty$, can be predicted by contour integration methods (Appendix 1). As one expects, they are given by equations (24) and (25) with $(b_1)_{-\infty} = \Delta$ and $(v_1)_{-\infty} = 0$. The value of b in, and near, the resultant shock layer was calculated, with the aid of a computer, for various non-large values of T. In this work, $Z = 1{\cdot}5$, $m_1^2 = 6$ and $Q = 1{\cdot}1335$. The value of Q used is appropriate to a shock, density ratio 1·5, propagating through a perfect, monatomic, fully ionized gas. The profiles obtained are shown in figure 1. Two diffusing Alfvén waves of the strength predicted by infinite conductivity theory (Akhiezer *et al.*) can be seen emerging and propagating away from the shock. Behind them lies the asymptotic steady-state profile.

3.2. *An asymptotic solution for the trans-Alfvénic case* $(Z > m_1^2 > 1)$

In this section an asymptotic solution, applicable for large T, is obtained for a slightly more general initial disturbance.

$$b_1(x, 0) = f_1, b_2(x, 0) = f_2, v_1(x, 0) = g_1 \quad \text{and} \quad v_2(x, 0) = g_2$$

where f_1, f_2, g_1 and g_2 are numerical constants.

There is a pole of order 2 at $s = 0$ in the integrand of the integral expression for b. If the shock had been super-Alfvénic, as in § 3.2, or sub-Alfvénic, there would have been a pole of order 1 at $s = 0$. This corresponds to the fact that there is now a natural singularity in the equations, i.e. there exist non-zero solutions to equations (32) and (33) when $s = b(x, 0) = v(x, 0) = 0$. This is the source of the indeterminacy in the steady-state shock profile discussed in § 2.4.

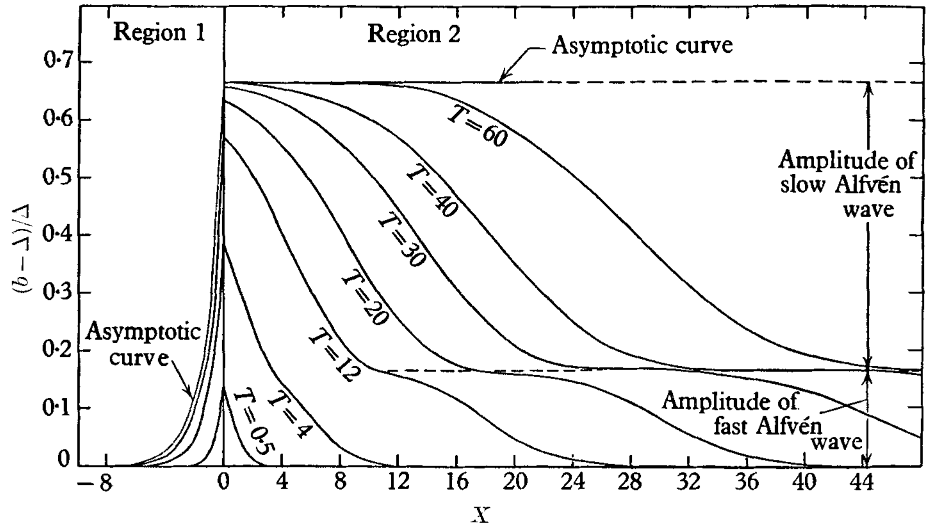

FIGURE 1. The relative increase in transverse magnetic field near the super-Alfvénic normal shock.

For this problem

$$b_1 = f_1 + \frac{1}{2\pi i}\int_{\phi - i\infty}^{\phi + i\infty} A_{11} \exp(\beta_{11} X + sT)\, ds,$$

where

$$A_{11} = \frac{\beta_{22}\beta_{23}\{(Zf_1 - f_2) - (Zg_1 - g_2)\} + s(\beta_{22} + \beta_{23})(f_1 - f_2) + (Z-1)\, sf_1\beta_{11}}{(Z\beta_{11} - (Z-1) - \beta_{23})\beta_{11} + \beta_{22}(Z\beta_{23} Q - \beta_{11})} \frac{\beta_{11}}{s^2}$$

and $\phi = \mathrm{Re}\, s$;

β_{11} is that root of equation (34) which has a positive real part ($i = 1$); β_{22} and β_{23} are those roots which have a negative real part ($i = 2$).

As $s \to 0$,

$$\beta_{11} \sim \left(\frac{m_1^2 - 1}{m_1^2}\right) + \left(\frac{m_1^2 + 1}{m_1^2 - 1}\right) Qs + O(Q^2 s^2) \quad \text{and} \quad A_{11} \sim (B/s^2) + (A/s) + O(1),$$

where A and B are defined by equations (35) to (39). Hence the contribution to the integral from the pole at $s = 0$ is

$$(A + \{(m_1^2 + 1)/(m_1^2 - 1)\}\, BQX + BT) \exp\{(m_1^2 - 1)\, X/m_1^2\}.$$

The path of integration has been closed to the left ($\mathrm{Re}\, s < 0$) along the infinite semi-circle in the s-plane. If it can be shown† that all other contributions tend to zero as $T \to \infty$, the above expression gives the asymptotic form of b_1.

The full expressions for A and B are

$$B = K_2^{-1}[(Z-1)f_1 + \{(Z - m_1^2)/m_1^2(m_2 + 1)\}\{m_2(f_2 - f_1) - m_2^2(Zg_1 - g_2)\}], \quad (35)$$

$$A = K_2^{-1}[K_3 B + K_5], \quad (36)$$

† See Appendix 1.

where $$K_2 = Z\left(\frac{m_1^2+1}{m_1^2-1}\right)Q+\frac{Z+m_1^2}{Z-m_1^2}-\frac{\{Q(Z-m_1^2)+(m_1^2-1)\}}{(m_1^2-1)(m_2+1)}, \tag{36}$$

$$K_3 = \frac{Zm_1^4(m_1^2+3)Q^2}{(m_1^2-1)^3}-\frac{K_1\{Q(Z-m_1^2)+(m_1^2-1)\}}{(m_1^2-1)(m_2+1)}+\frac{m_1^4(3Z+m_1^2)}{(Z-m_1^2)^3}, \tag{37}$$

$$K_1 = \frac{-m_2^3}{2(m_2+1)^2}+\frac{2m_1^4Q}{(m_1^2-1)^2}+\frac{2m_1^4(Z-1)Q}{[m_1^2(Q-1)-(QZ-1)](m_1^2-1)(Z-m_1^2)}, \tag{38}$$

and $$K_5 = (f_2-f_1)\left\{\left(\frac{Z+m_1^2}{Z-m_1^2}\right)\frac{m_2}{(m_2+1)}-\frac{(Z-m_1^2)m_2}{2Z(m_2+1)^3}\right\}$$
$$-m_2^2(Zg_1-g_2)\left\{\frac{(Z-m_1^2)m_2}{2Z(m_2+1)^3}+\frac{2m_1^2}{(Z-m_1^2)(m_2+1)}\right\}. \tag{39}$$

By a similar procedure we find that, as $T\to\infty$,

$$b_2\to f_2+C+\left\{A-f_2+f_1-C-\frac{(Z+m_1^2)}{(Z-m_1^2)}BX+BT\right\}\exp\{-(Z-m_1^2)X/m_1^2\}. \tag{40}$$

C is the strength of the only Alfvén wave (a right travelling one) that the shock can emit

$$C = K_2^{-1}\left[Z\left\{\frac{(QZ-1)-m_1^2(Q-1)}{(m_1^2-1)(Z-m_1^2)}\right\}\left\{(Z-1)f_1+\left(\frac{Z-m_1^2}{m_1^2}\right)(f_2-f_1)\right\}\right.$$
$$\left.-\left\{\left(\frac{m_1^2+Z}{Z-m_1^2}\right)+\frac{Z(m_1^2+1)}{(m_1^2-1)}Q\right\}\left\{\frac{Z}{m_1^2}(f_2-f_1)-g_2+Zg_1\right\}\right]\left(\frac{Z}{m_1^2}+\frac{1}{m_2}\right)^{-1}. \tag{41}$$

It is worth while noting that an easier way to find B and C is to try solutions of the type

$$b_1 = (f_1+G_1(X)+BT)\exp\{(m_1^2-1)X/m_1^2\},$$
$$b_2 = f_2+C+(G_2(X)+BT)\exp\{-(Z-m_1^2)X/m_1^2\}.$$

G_1, G_2 are found to be of the same type as before and the values of B and C obtained are, of course, in agreement with the above values. However, the value of A could not be determined by this method.

We shall refer to the above values of A, B and C, as

$$A(f_1,g_1,f_2,g_2),\quad B(f_1,g_1,f_2,g_2)\quad\text{and}\quad C(f_1,g_1,f_2,g_2),$$

respectively. The solution for b, in this problem, we shall refer to as $b(f_1,g_1,f_2,g_2)$. A can be looked upon as the memory of the shock.

3.3. *Computations of an initial value problem in a trans-Alfvénic case*

The initial disturbance is the same as in § 3.2, i.e. $g_1 = g_2 = 0$; $f_1 = f_2 = \Delta$ (const.) $Z = 3$, $m_1^2 = 2$ and $Q = 1$. It was not expedient to set the value of Q in the manner discussed in § 3.1. This was for computational reasons. The value of unity is, however, just as realistic.

The computed profiles of $(b/\Delta-1)$ are plotted, for various values of T, in figure 2. The variation of $(b/\Delta)-1$, at $x=0$, with T, is given in figure 3. The computed values of b/Δ have an absolute error of less than 10^{-3} for $T\leqslant 80$ and less than $10^{-3}\exp\frac{1}{8}(T-80)$ for larger times. The amplitude of the emergent Alfvén wave and the values of A and B are in agreement with the results of §3.3, which give

$$C/f_1 = 0{\cdot}3414, \quad A/f_1 = 1{\cdot}2512 \quad \text{and} \quad B/f_1 = 0{\cdot}15505.$$

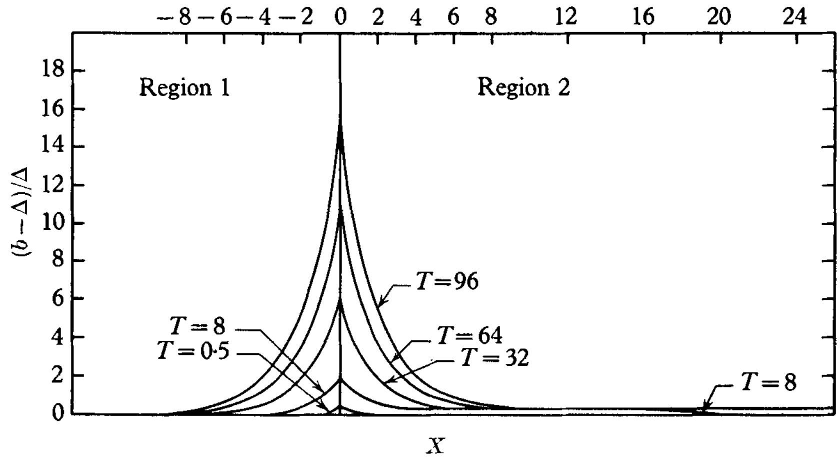

FIGURE 2. The relative increase in transverse magnetic field near the trans-Alfvénic normal shock.

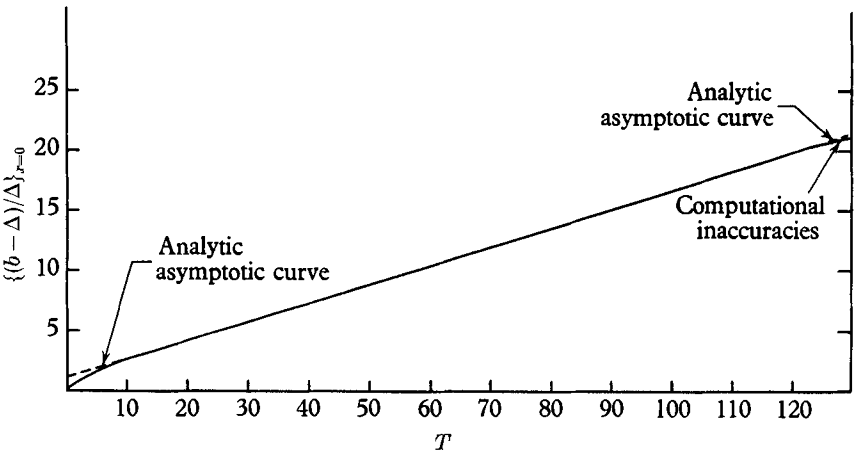

FIGURE 3. The variation of the relative increase of transverse magnetic field at the shock with time.

It is noticeable that the Alfvén wave is formed much more quickly than the corresponding wave in §3.2, though the latter will eventually travel more quickly than it. As soon as the wave has been formed, i.e. ejected from the shock region, the shock profile is very near the theoretical asymptotic form.

Some comparisons between the computed profiles and the relevant theoretical asymptotic ones are made below in table 1. The figures in brackets correspond to the theoretical asymptotic profiles.

$T \setminus X$	−12	−8	−4	−2	0
16	1·0004 (0·9954)	1·0122 (1·0002)	1·2461 (1·2533)	1·9699 (2·0307)	4·5579 (4·7320)
32	1·0035 (1·0016)	1·0488 (1·0456)	1·5827 (1·5890)	2·9174 (2·9433)	7·1457 (7·2128)
64	1·0141 (1·0139)	1·1367 (1·1365)	2·2582 (2·2605)	4·7623 (4·7686)	12·161 (12·174)

$T \setminus X$	1	2	4	8	16
16	2·8606 (2·9277)	2·0262 (2·0183)	1·4558 (1·3806)	1·3369 (1·2699)	1·3330 (1·3384)
32	4·4078 (4·4324)	2·9353 (2·9310)	1·7472 (1·7163)	1·3592 (1·3353)	1·3390 (1·3392)
64	7·4362 (7·4417)	4·7581 (4·7562)	2·3955 (2·3878)	1·4327 (1·4262)	1·3413 (1·3409)

TABLE 1. Some comparisons between the theoretical asymptotic profiles and the computed shock profiles of the relative transverse field

4. The resolution of an arbitrary initial disturbance

The results of § 3 showed that the asymptotic shock profiles of b are indeed predicted by the analytical method† used. It is assumed this is so for the problems discussed in this section. The shock is trans-Alfvénic.

4.1. *Two more simple, initial-value problems*

$$\text{(1)}\quad \left.\begin{array}{l} b_1 = \Delta(\text{const.}),\ \ v_1 = \delta(\text{const.}),\ \ X+L<0 \\ b = v = 0,\ \ X+L>0 \end{array}\right\} \quad \text{at} \quad T=0,$$

where L is any positive constant. The asymptotic solution† for b_1 is

$$b_1 \sim b_1(\Delta,\delta,0,0) - \frac{ZQ}{K_2}(\Delta + m_2\delta)\,L\exp(1-m_1^{-2})\,X, \tag{42}$$

$$\text{(2)}\quad \left.\begin{array}{l} b = v = 0,\ \ X-N<0 \\ b_2 = \phi(\text{const.}),\ \ v_2 = \psi(\text{const.}),\ \ X-N>0 \end{array}\right\} \quad \text{at} \quad T=0,$$

where N is a positive constant. In this case, as $T \to \infty$†

$$b_1 \sim b_1(0,0,\phi,\psi) - K_2^{-1}N(\phi + m_2\psi)\exp(1-m_1^{-2})\,X. \tag{43}$$

4.2. *An arbitrary initial disturbance*

If, at $T = 0$, b and v tend to limiting values as $|X| \to \infty$ in such a way that

$$\int_0^{\pm\infty} (b-(b)_{\pm\infty})\,dX \quad \text{and} \quad \int_0^{\pm\infty} (v-(v)_{\pm\infty})\,dX$$

all exist, the results of the last section may be used to find the asymptotic solution for b_1.

† See Appendix 1.

The change in A, the shock memory, due to the perturbation pictured in figure 4*a* is obviously given by

$$\delta A = \delta L\left(-\frac{d}{dL}\right)\left(-\frac{ZQ}{K_2}(\Delta+m_2\delta)\,L\right) = \delta L\frac{ZQ}{K_2}(\Delta+m_2\delta).$$

Similarly the change in A due to the initial state shown in figure 4*b* is

$$\delta A = \delta N\,K_2^{-1}(\phi+m_2\psi).$$

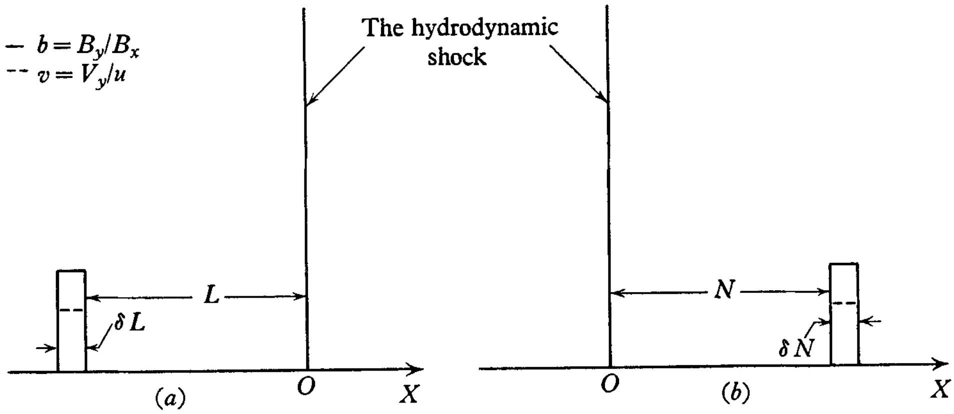

FIGURE 4. Initial disturbances of two possible types.

Hence the asymptotic solution, for any initial disturbance of the stated type, is

$$b_1 = b_1\{(b_1)_{-\infty}, (v_1)_{-\infty}, (b_2)_{+\infty}, (v_2)_{+\infty}\} + A'\exp(1-m_1^{-2})\,X, \tag{44}$$

where

$$A' = \frac{ZQ}{K_2}\int_{-\infty}^{0}\{b_1(X,0)-(b_1)_{-\infty}\}\,dX + \frac{Zm_2Q}{K_2}\int_{-\infty}^{0}\{v_1(X,0)-(v_1)_{-\infty}\}\,dX$$
$$+\frac{1}{K_2}\int_{0}^{\infty}\{b_2(X,0)-(b_2)_{+\infty}\}\,dX + \frac{m_2}{K_2}\int_{0}^{\infty}\{v_2(X,0)-(v_2)_{+\infty}\}\,dX. \tag{45}$$

Because this is a linear problem, the disturbance has been treated as one of finite extent, which provides the term in A', superimposed on one of the type discussed in §3.2. If a steady state were reached inside the magnetohydrodynamic shock, i.e. $B\{(b_1)_{-\infty}, (v_1)_{-\infty}, (b_2)_{\infty}, (v_2)_{\infty}\} = 0$, the value of A, A_N (say), would be given by

$$A_N = A' + A((b_1)_{-\infty}, (v_1)_{-\infty}, (b_2)_{+\infty}, (v_2)_{+\infty}). \tag{46}$$

This is the integral equation which resolves the anomaly mentioned in §2.3. It means that the steady-state shock structure can be fixed if the perturbed flow field at any previous time is known.

Obviously a similar procedure could be carried out for a quite arbitrary initial state. For large T, the change in the memory of the shock could be reasonably estimated by an integral expression similar to that of equation (46). The part of the disturbance which would have affected A, at any time, is that part which would have reached the shock via the diffusing Alfvén waves.

4.3. *An interpretation of the results*

Equation (5) may be rewritten as follows:

$$\left.\begin{aligned}\frac{\partial}{\partial t}\{B_q-(B_q)_{-\infty}\}+\frac{\partial(V_{x_1}B_{q_1})}{\partial x}-\frac{\partial}{\partial x}\left(\lambda_1\frac{\partial B_{q_1}}{\partial x}\right)-B_x\frac{\partial V_{q_1}}{\partial x}=0 \quad (x<0),\\ \frac{\partial}{\partial t}\{B_{q_2}-(B_q)_{+\infty}\}+\frac{\partial}{\partial x}(V_{x_2}B_{q_2})-\frac{\partial}{\partial x}\left(\lambda_2\frac{\partial B_{q_2}}{\partial x}\right)-B_x\frac{\partial V_{q_2}}{\partial x}=0 \quad (x>0).\end{aligned}\right\}$$

Let us define $\Lambda=\Lambda_1+\Lambda_2$ and $\chi=\chi_1+\chi_2$, where

$$\Lambda_1=\int_{-\infty}^{0}(B_q-(B_q)_{-\infty})\,dx \quad\text{and}\quad \Lambda_2=\int_{0}^{+\infty}(B_q-(B_q)_{+\infty})\,dx,$$

$$\chi_1=\int_{-\infty}^{0}(v_q-(v_q)_{-\infty})\,dx \quad\text{and}\quad \chi_2=\int_{0}^{\infty}(v_q-(v_q)_{+\infty})\,dx, \quad v_{qi}=V_{qi}/u_i.$$

It follows, with the aid of the boundary condition given by equation (12), that

$$\frac{d\Lambda}{dt}=\left[\lambda\frac{\partial B_q}{\partial x}-V_xB_q+B_xV_q\right]_{-\infty}^{+\infty}.$$

Similarly from equation (16) and boundary condition (13)

$$\frac{d\chi}{dt}=\left[\frac{B_x}{\mu\rho u}B_q-V_q\right]_{-\infty}^{+\infty}.$$

Let us apply these results to the following two cases:

(i) A disturbance of finite extent, i.e. b_q and v_q are identically zero far from the shock. In this case $\Lambda=$ const. and $\chi=$ const. Hence both the flux of transverse magnetic field and of v_q are conserved.

(ii) The disturbance considered in §3.2, i.e. $b_i(x,0)=f_i$ and $v_i(x,0)=g_i$, where f_i and g_i are constants ($i=1,2$). For this problem

$$\Lambda=B_x\{u_1(f_1-g_1)-u_2(f_2-g_2)\}t \quad\text{and}\quad \chi=\{u_1(g_1-m_1^{-2}f_1)-u_2(g_2-Zm_1^{-2}f_2)\}t.$$

Thus, in general, both fluxes grow linearly with time. These results are true whatever the values of m_1^2, Z and Q, *and whether or not* λ *is zero.* In an Alfvén wave the fluxes of b_q and v_q are linearly connected and hence these two conditions cannot, in general, be satisfied simultaneously by one wave. Thus if the normal shock is super-Alfvénic or sub-Alfvénic these two conservation-type conditions determine the separate fluxes contained in the two outward travelling Alfvén waves.

For the trans-Alfvénic normal shock there is, of course, only one outward travelling wave and at least one diffusivity must be taken into account. Let us take the magnetic diffusivity λ, as the dominant one. We have now given the trans-Alfvénic normal shock, but no other kind of normal shock,† the ability to store flux within itself and consequently the two conservation-type conditions can be satisfied simultaneously. The reader is reminded that if steady states exist

† See §2.4.

immediately outside the shock region, the flux of b_q and v_q are linearly dependent, at any time, on one parameter.† Obviously the results of §4.2 can be obtained in the following way: (i) calculate the change in flux of v_q and b_q at the shock due to the disturbance pictured in figure 4a in terms of the change in transverse magnetic field at the shock δA; (ii) add these to the relevant fluxes in the outgoing wave and satisfy the two conservation conditions; (iii) repeat for the disturbance pictured in figure 4b and proceed as before.

Let us now consider what happens in the two limiting cases of the trans-Alfvénic normal shock, i.e. the null switch-on and null switch-off shocks, if the initial disturbance is either of the types pictured in figure 4. The two conservation conditions are still valid. A wave is propagated away from the shock and flux is left at the shock exactly as for the trans-Alfvénic normal shock. However, because convection and propagation are in exact balance on one side of the shock, diffusion has become the dominant physical process. If no further disturbances are incident, the flux at the shock will simply diffuse into the region in which $m = 1$. An arbitrary initial disturbance will, in general, deposit flux at either of these shocks at a greater rate than the rate at which it can be diffused away. Consequently both null switch-on and null switch-off shocks are unstable. However, the build-up of transverse magnetic field at the shock is not so rapid as for a trans-Alfvénic normal shock.

5. Conclusions

The trans-Alfvénic normal shock is unstable to transverse perturbations in the magnetic field and velocity. When a small packet of perturbed flow reaches and is propagated away from the shock region via the diffusing Alfvén wave, the change in transverse field at the shock due to this packet is linearly dependent on both the flux of magnetic field and the flux of transverse velocity in the packet at the initial time. Hence there exist particular disturbances which do not break up the shock. If the shock encounters an Alfvén wave in the form of a weak diffusing current sheet, the magnitude of the transverse field at the shock eventually grows linearly with time, an unstable situation. It is the intention of the author to investigate the non-linear problem which arises in this case. If the initial small disturbances are such that a steady state is reached (unlikely in general), the shock profiles of the transverse quantities are fixed by introducing the additional integral condition of equation (46). Thus the puzzling arbitrariness of the steady state shock profiles of V_q and B_q is resolved, $q = (y, z)$.

An important conclusion of this work is that, in distinct contrast to classical theories, the unsteady processes which may be taking place within the trans-Alfvénic magnetohydrodynamic shock cannot, in general, be neglected in the derivation of jump conditions across the shock. Hence, in general, neither the jump in transverse momentum plus Maxwell stress nor the change in the transverse components of the electric field is zero across the magnetohydrodynamic shock.

Null switch-on and switch-off shocks are also unstable to the general perturbation field, though there exist disturbances which do not destroy them. Any

† See §3.2.

transverse magnetic field or transverse velocity being built up at the shock will tend to diffuse into the region in which $m = 1$. The only steady state possible at, or near one of these shocks has zero transverse magnetic field and zero transverse velocity.

I am grateful to Dr J. A. Shercliff for the many stimulating discussions which we have had. This work was carried out while the author was a holder of a Research Studentship from the Department of Scientific and Industrial Research.

Appendix 1

All the theoretical asymptotic forms of b_1 mentioned in this paper were obtained by evaluating the residue of the integrand, in the integral expression for b_1, at $s = 0$. The asymptotic form of b_2 is easily obtained from that of b_1. In general

$$b_1 = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \sum_{p=1}^{3} \epsilon_p \exp(B_{1p}X_1 + sT)\,ds = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \epsilon\, ds \quad \text{(say)},$$

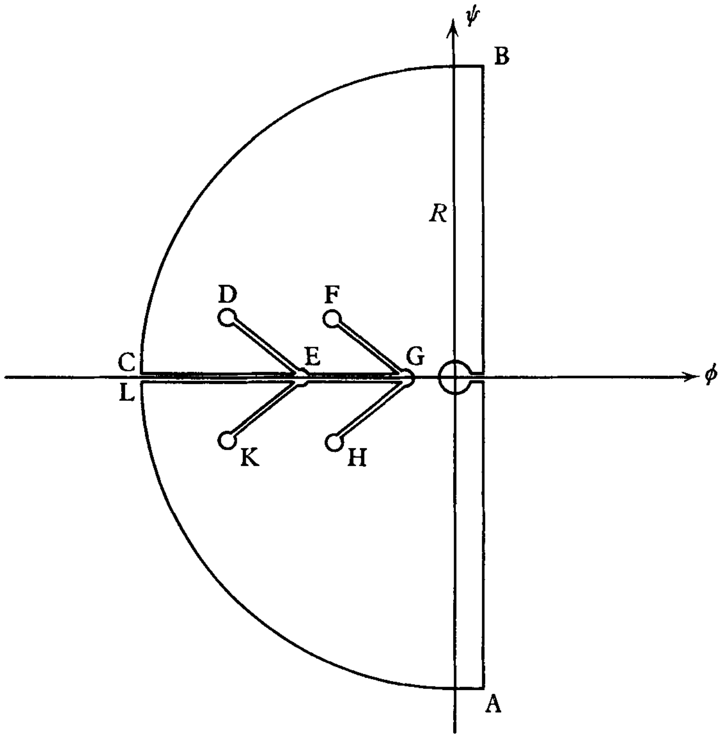

FIGURE 5. The closed path of integration in the complex s-plane.

where ϵ_p is a quotient of two simple algebraic functions of the roots of (34) ($i = 1, 2$), e.g.

$$\epsilon_1 = \frac{\beta_{11}\beta_{23} - Z\beta_{12}\beta_{22}}{(\beta_{11} + Qs)(\beta_{21} + (Z/m_1^2) + s)}.$$

Branch cuts in the s-plane are necessary to make a root of equation (34) analytic. All the branch points are in the half-plane $\operatorname{Re} s < 0$ for $m_1^2 \neq Z$ or 1. These are

D, E, F, G, H and K. There is also a branch point at infinity. The contour Γ is therefore closed as shown in figure 5. It is relatively easy to show that, in the problems considered,

$$\int_{BC} \epsilon\, ds \quad \text{and} \quad \int_{LA} \epsilon\, ds \to 0 \quad \text{as} \quad R \to \infty.$$

Furthermore
$$\int_{CD\ldots KL} \epsilon\, ds \to 0 \quad \text{as} \quad T \to \infty.$$

Thus, provided that there exist no poles of the integrand in $\operatorname{Re} s \geqslant 0$ other than at $s = 0$, the asymptotic solution, as $T \to \infty$, is given by $2\pi i \times$ (residue at $s = 0$). In the initial value problem considered in §§3.1 and 3.3

$$\epsilon = \mathrm{F}(\beta\text{'s}, m_1, Z, Q, s)/s\mathrm{G}(\beta\text{'s}, m_1, Z, Q, s)$$

where F and G are of the stated type. F must always be finite or zero. The two sets of computations carried out lend evidence to the idea that G has no zeros in $\operatorname{Re} s \geqslant 0$, other than at $s = 0$. This is a sufficient condition that the analytic work of §3 be valid and that an incident Alfvén wave in the form of a diffusing current sheet will produce the same qualitative effects. The analytic work of §4 is also subject to there being no contributions from poles of the relevant integrand in $\operatorname{Re} s \geqslant 0$, other than at $s = 0$.

As $m_1^2 \to 1$ (or Z), the points E (or G) $\to 0$ and consequently the nearer m_1^2 is to 1 (or Z) the longer it takes for the asymptotic limit to be reached.

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